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LETTER TO THE EDITOR

Parabosons, Virasoro-type algebras and their deformations

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Abstract. Two-parameter (p, q) deformation of a parabosonic algebra underlying the two-particle Calogero model is considered. The Fock-space representation, Bargmann–Fock representation, coherent states, spectrum-generating algebra and a hidden supersymmetry are discussed. The paraboson algebra induces a centreless Virasoro-type algebra with a doubling of the space of generators. A two-parameter deformation of this Virasoro-type algebra is studied.

Study of generalized oscillator systems is becoming an interesting subject pursued actively with a view to exploring possible new physical situations. Recently, the algebra

$$\begin{aligned} &[\beta, \beta^\dagger] = 1 + 2\nu\mathcal{K} \qquad \mathcal{K} = (-1)^{\mathcal{N}} \quad \nu \in \mathbb{R} \\ &[\mathcal{N}, \beta] = -\beta \qquad [\mathcal{N}, \beta^\dagger] = \beta^\dagger \end{aligned} \tag{1}$$

has been introduced [1,2] and found [2] to underly the two-particle Calogero model [3]. Calling the algebra (1) a modified oscillator algebra and combining it with the idea of the well known q-deformation [4] of the oscillator algebra a new generalized oscillator algebra has now been obtained [5] with the motivation of exploiting it to construct new integrable systems. This new algebra [5] reads

$$\begin{aligned} &aa^\dagger - qa^\dagger a = (1 + 2\nu K)q^{-N} \qquad K = (-1)^N \quad q, \nu \in \mathbb{R} \\ &[N, a] = -a \qquad [N, a^\dagger] = a^\dagger . \end{aligned} \tag{2}$$

When the parameter 2ν is an integer $\geqslant 1$ it has been shown [6] that the algebra (1) represents a single-mode oscillator obeying para-Bose statistics of order $2\nu+1$. Hence, we shall refer to the algebra (1), in general for any $\nu \in \mathbb{R}$, as a parabosonic algebra. Since a two-parameter generalization of the algebra (2) would provide more flexibility in the context of model building we consider here such a generalization by combining the ideas of modification (1) and (p, q)-deformation [7, 8] of the oscillator algebra. For the resulting (p, q)-deformation of (1) we discuss, following [5], the corresponding Fock-space, spectrum-generating algebra and the hidden supersymmetry engendered by the presence of the so-called Klein operator K; the Bargmann–Fock realization, and coherent states are also discussed.

Quantum groups and algebras with multiple deformation parameters have been studied from the point of view of construction, representation and applicability in concrete physical models [7–12]. Considering the quantum algebra $U_{p,q}(gl(2))$, with two deformation parameters (p, q), it has been noted [10] that although the algebraic structure can be mapped

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on to a $U_Q(gl(2))$ with a single deformation parameter $Q = \sqrt{pq}$, the parameters p and q are genuinely independent as a dependence of the co-multiplication rules and the universal $\mathcal{R}$-matrix on $\lambda = \sqrt{p/q}$ persists. The (p,q)-oscillator algebra [7, 8]

$$aa^\dagger - qa^\dagger a = p^{-N} \qquad [N,a] = -a \qquad \left[N,a^\dagger\right] = a^\dagger \tag{3}$$

is obtained as a contraction limit of $U_{p,q}(gl(2))$ and generates [7] a Jordan–Schwinger-type realization thereof. The algebra (3) has been used [13] in the construction of a (p,q)-deformed Virasoro algebra in which the co-algebraic structure of the functional generator exhibits the genuine two-parametric nature of the deformation.

In the spirit of the above discussion, let us first consider a two-parameter deformation of the parabosonic algebra (1) given by

$$\begin{aligned} &aa^\dagger - qa^\dagger a = (1+2\nu K)p^{-N} \qquad K=(-1)^N \quad p,q,\nu\in\mathbb{R} \\ &[N,a] = -a \qquad [N,a^\dagger] = a^\dagger \end{aligned} \tag{4}$$

which reduces to (1) when $p = q = 1$. The algebra (2) considered in [5] is the special case of (4) corresponding to the choice $p = q$.

As has often been recognized in the literature, any generalized bosonic oscillator algebra can be presented in the form

$$a^\dagger a = \phi(N) \qquad aa^\dagger = \phi(N+1) \qquad [N,a] = -a \qquad \left[N,a^\dagger\right] = a^\dagger \tag{5}$$

where $\phi(N)$ characterizes the system and is a real non-negative function ($\phi(n) \geqslant 0$ for any $n \geqslant 0$). In the case of the system (4)

$$\phi(N) = [N]_{p,q} + 2\nu[N]_{-p,q} \quad \text{with} \quad [X]_{p,q} = \frac{q^X - p^{-X}}{q - p^{-1}}\,. \tag{6}$$

Now, $\phi(0) = 0$ and we shall require $\phi(n) > 0$ for any $n > 0$. Then, the condition on the parameters $p,q,\nu \in \mathbb{R}$ is as follows. With both $p,q > 0$ or < 0 and $\chi = \left(q - p^{-1}\right)/\left(q + p^{-1}\right)$

$$2\nu > -1 \quad \text{for} \quad pq > 1 \qquad -\chi^{-1} > 2\nu > -1 \quad \text{for} \quad 0 < pq < 1\,. \tag{7}$$

The Fock representation of the algebra (4) is easily constructed by the action of monomials in $a^\dagger$ on the vacuum state $|0\rangle$ assumed to be unique and defined by $a|0\rangle = 0$, $N|0\rangle = 0$. The complete orthonormal set of eigenstates of the number operator N $\{|n\rangle \mid n = 0,1,2,\ldots\}$ satisfying

$$\begin{aligned} &N|n\rangle = n|n\rangle \qquad \phi(N)|n\rangle = \phi(n)|n\rangle \\ &a|n\rangle = \sqrt{\phi(n)}\,|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{\phi(n+1)}\,|n+1\rangle \end{aligned} \tag{8}$$

are given by

$$|n\rangle = \frac{1}{\sqrt{\phi(n)!}} a^{\dagger n}|0\rangle \qquad \phi(n)! = \phi(n)\phi(n-1)\ldots\phi(1) \quad \phi(0)! = 1\,. \tag{9}$$

The above construction may be verified using the identity

$$aa^{\dagger n} = q^n a^{\dagger n} a + a^{\dagger(n-1)} p^{-N}\left([n]_{p,q} + 2\nu[n]_{-p,q}K\right) \qquad \text{for} \quad n \geqslant 1\,. \tag{10}$$

The boson realization of the algebra (4), following from (8) is given by

$$a = b\sqrt{\frac{\phi(N)}{N}} \qquad a^\dagger = \sqrt{\frac{\phi(N)}{N}}\,b^\dagger \qquad N = b^\dagger b\,. \tag{11}$$

The Bargmann–Fock realization of the algebra (4) is obtained by the map

$$a^\dagger = z \qquad N = z\frac{\partial}{\partial z} \qquad a = \frac{1}{z}\phi\left(z\frac{\partial}{\partial z}\right) = D_{p,q}(z) + 2\nu D_{-p,q}(z)$$
$$\text{with} \quad D_{p,q}(z) = \frac{1}{z}\left[z\frac{\partial}{\partial z}\right]_{p,q} \tag{12}$$

in the space of analytic functions of the complex variable z. The inner product structure which makes a and $a^\dagger$, in the representation (12), Hermitian conjugates is

$$(f, g) = \left(f\left(D_{p,q}(z) + 2\nu D_{-p,q}(z)\right) g(z)\right)|_{z=0} \, . \tag{13}$$

The set of functions $\left\{z^n/\sqrt{\phi(n)!}\,\middle|\, n = 0, 1, 2, \ldots\right\}$ is seen to form an orthonormal basis with respect to the inner product (13). In the undeformed limit ($p = q = 1$) the representation (12) assumes the form

$$\beta = \frac{\partial}{\partial z} + \frac{\nu}{z}\left\{1 - (-1)^{z\partial/\partial z}\right\} \qquad \beta^\dagger = z \qquad \mathcal{N} = z\frac{\partial}{\partial z} \, . \tag{14}$$

For the paraboson algebra (1) the number operator is known [6] to be given by $\mathcal{N} = \frac{1}{2}\left(\beta\beta^\dagger + \beta^\dagger\beta\right) - \frac{1}{2}(2\nu + 1)$. The number operator for the general (p, q)-deformed parabosonic algebra (4) can be obtained using a general procedure outlined in [14]. In the special case $p = 1$ an explicit expression for N can also be obtained by inverting the relations in (5) and (6):

$$N = \frac{1}{\ln q}\ln\left(\frac{(q-1)\left(aa^\dagger + a^\dagger a\right) + 2}{(q+1) + 2\nu(q-1)}\right) . \tag{15}$$

The construction of coherent states of the (p, q)-paraboson (4) is most easily done following a simple technique [6, 15] applicable to any generalized boson oscillator. Let

$$A = a\frac{N}{\phi(N)} \qquad A^\dagger = \frac{N}{\phi(N)}a^\dagger \tag{16}$$

such that

$$\left[a, A^\dagger\right] = 1 \qquad \left[A, a^\dagger\right] = 1 \qquad [N, A] = -A \qquad \left[N, A^\dagger\right] = A^\dagger \, . \tag{17}$$

Then the coherent state $|z\rangle$ obeying $a|z\rangle = z|z\rangle$ is obtained by taking $|z\rangle \sim \exp\left(zA^\dagger\right)|0\rangle$. The normalized $|z\rangle$ is

$$|z\rangle = \left\{\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\phi(n)!}\right\}^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\phi(n)!}}|n\rangle \, . \tag{18}$$

In the undeformed limit ($p = q = 1$) we get the known result [6, 16] for the paraboson. It may also be noted that $|z\rangle' \sim \exp\left(za^\dagger\right)|0\rangle$ gives an eigenstate of A as has been pointed out already [6, 15].

The spectrum-generating algebra for the (p, q)-parabosonic system (4) is a $U_{p^2,q^2}(su(1,1) \oplus u(1))$ in which the non-zero central term is essential for maintaining the two-parametric nature of the deformation. Our discussion of this spectrum-generating algebra closely follows [5]. The generators of this dynamical algebra, apart from a central element Z, are as follows:

$$B_0 = \frac{1}{2}\left(N + \frac{1}{2}\right) \qquad B_+ = \xi a^{\dagger 2} \qquad B_- = -\xi a^2$$
$$\text{with} \quad \xi = \left(p + q^{-1}\right)^{-1}(1 + 2\nu\chi)^{-3/4}(1 - 2\nu\chi)^{-1/4} \, . \tag{19}$$

These generators obey the commutation relations

$$[Z, \cdot] = 0 \qquad \left[B_0, B_\pm\right] = \pm B_\pm \qquad B_+B_- - (p/q)^2\, B_-B_+ = \frac{\eta^2 - \zeta^{-2}}{q^2 - p^{-2}}$$
$$\text{with} \quad \eta = \mu q^{2B_0} \quad \zeta = \mu p^{2B_0} \quad \mu = \left(\frac{1+2\nu\chi}{1-2\nu\chi}\right)^{1/4}. \tag{20}$$

The expression $\left(\eta^2 - \zeta^{-2}\right) / \left(q^2 - p^{-2}\right)$, which becomes $N + \nu + \frac{1}{2}$ in the limit $p = q = 1$, may be identified with the Hamiltonian of the system (4). The co-product rules for the generators preserving the algebraic structure (20) are

$$\begin{aligned} &\Delta(Z) = Z \otimes \mathbf{1} + \mathbf{1} \otimes Z \qquad \Delta\left(B_0\right) = B_0 \otimes \mathbf{1} + \mathbf{1} \otimes B_0 \\ &\Delta\left(B_\pm\right) = B_\pm \otimes \eta\lambda^{\pm Z} + \zeta^{-1}\lambda^{\mp Z} \otimes B_\pm\,. \end{aligned} \tag{21}$$

In an irreducible representation the central element Z assumes an arbitrary imaginary constant value. It plays a key role in imparting the two-parametric nature to the deformations as is evident from the fact that if we take $Z = 0$, then both the algebraic and co-algebraic structures may be simultaneously transformed to depend on the single parameter Q. To see this, we observe the following. The map

$$\tilde{B}_0 = B_0 \qquad \tilde{B}_\pm = B_\pm \lambda^{2B_0 \pm 1} \qquad \tilde{Z} = Z \tag{22}$$

reduces the algebra (20) to the standard form [5]

$$\begin{aligned} &\left[\tilde{Z}, \cdot\right] = 0 \qquad \left[\tilde{B}_0, \tilde{B}_\pm\right] = \pm\tilde{B}_\pm \\ &\left[\tilde{B}_+, \tilde{B}_-\right] = \frac{\tilde{\eta}^2 - \tilde{\eta}^{-2}}{Q^2 - Q^{-2}} \qquad \text{with} \quad \tilde{\eta} = \mu Q^{2\tilde{B}_0}. \end{aligned} \tag{23}$$

Note that μ depends only on Q. However, when $\tilde{Z} \neq 0$ the induced co-products for the generators depend on both the deformation parameters:

$$\begin{aligned} &\Delta\left(\tilde{Z}\right) = \tilde{Z} \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{Z} \qquad \Delta\left(\tilde{B}_0\right) = \tilde{B}_0 \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{B}_0 \\ &\Delta\left(\tilde{B}_\pm\right) = \tilde{B}_\pm \otimes \tilde{\eta}\lambda^{\pm\tilde{Z}} + \tilde{\eta}^{-1}\lambda^{\mp\tilde{Z}} \otimes \tilde{B}_\pm\,. \end{aligned} \tag{24}$$

In the limit $\lambda = 1$ we recover the standard result for $U_{Q^2}(su(1,1))$ [5]. The Casimir operator for the above spectrum-generating algebra $U_{p^2,q^2}(su(1,1) \oplus u(1))$ is given by

$$C = \eta^{-1}\zeta\left(B_+B_- + \frac{\left(\eta p^{-1} - \zeta^{-1}q\right)^2 - \eta\zeta^{-1}\left(p^{-1} - q\right)^2}{\left(q^2 - p^{-2}\right)^2}\right). \tag{25}$$

From the map (22) it follows, however, that the operator C is the same as the Casimir operator $\mathcal{C}$ for $U_{Q^2}(su(1,1))$ [5] apart from a numerical scale factor:

$$C = \lambda^2\mathcal{C} \quad \text{where} \quad \mathcal{C} = \tilde{B}_+\tilde{B}_- + \frac{\left(\tilde{\eta}Q^{-1} - \tilde{\eta}^{-1}Q\right)^2 - \left(Q^{-1} - Q\right)^2}{\left(Q^2 - Q^{-2}\right)^2}. \tag{26}$$

It has been noted [5] that the presence of the operator K induces an unbroken supersymmetry in the Fock space corresponding to the q-deformed parabosonic algebra (2). This property persists for the (p, q)-deformed algebra (4). Additional freedom afforded by the two deformation parameters allows us to consider limiting cases which can be regarded, in the sense of [5], as modifications of various known oscillators. Following the analysis in [5], let us define a one-parameter family of Hermitian operators

$$O_\alpha(\omega) = e^{i\alpha}\omega^N a^\dagger P_- + e^{-i\alpha}a\omega^N P_+ \qquad 0 \leqslant \alpha < \pi \quad \text{with} \quad P_\pm = \tfrac{1}{2}(1 \pm K) \tag{27}$$

for a fixed $\omega \in \mathbb{R}$. Quite generally, the operators $\{O_\alpha(\omega)\}$ follow the algebra

$$\left\{O_\alpha(\omega), O_\beta(\omega)\right\} = 2\omega^{2(N+1)}\cos(\alpha-\beta)\left(aa^\dagger P_- + \omega^{-2}a^\dagger a P_+\right) . \tag{28}$$

For a choice

$$O_{\alpha=0}(\omega) = O_1(\omega) \qquad O_{\alpha=\pi/2}(\omega) = O_2(\omega) \tag{29}$$

we recover the supersymmetry algebra [5]

$$\left\{O_i(\omega), O_j(\omega)\right\} = 2\delta_{ij}H(\omega) \qquad i,j=1,2 \tag{30}$$

with a Hamiltonian

$$H(\omega) = \omega^{2(N+1)}\left(\phi(N+1)P_- + \omega^{-2}\phi(N)P_+\right) . \tag{31}$$

The Fock space shows features of unbroken supersymmetry:

$$\begin{aligned} H(\omega)\,|2n\rangle &= \omega^{4n}[2n]_{p,q}(1+2\nu\chi)\,|2n\rangle && n=0,1,2,\ldots \\ H(\omega)\,|2n-1\rangle &= \omega^{4n}[2n]_{p,q}(1+2\nu\chi)\,|2n-1\rangle && n=1,2,\ldots . \end{aligned} \tag{32}$$

Several special cases may be considered. For example, corresponding to a generalized oscillator with the algebra $aa^\dagger - a^\dagger a = q^N(1+2\nu K)$, i.e. algebra (4) with $q=1$ and $p=q^{-1}$, the choice $\omega=1$ leads to the Hamiltonian

$$H = \left(\frac{1}{1-q} + \frac{2\nu}{1+q}\right)\left\{1 - \frac{1}{2}\left((1+q)+(1-q)K\right)q^N\right\} . \tag{33}$$

Similarly, for an oscillator obeying the algebra $aa^\dagger - qa^\dagger a = (1+2\nu K)$, i.e. algebra (4) with $p=1$, the choice $\omega=1$ leads to the Hamiltonian

$$H = \left(\frac{1}{1-q} - \frac{2\nu}{1+q}\right)\left\{1 - \frac{1}{2}\left((1+q)+(1-q)K\right)q^N\right\} . \tag{34}$$

In the absence of modification, i.e. when $\nu=0$, the two Hamiltonians in (33) and (34) become equal corresponding to the fact that the generalized oscillators with $aa^\dagger - a^\dagger a = q^N$ and $aa^\dagger - qa^\dagger a = 1$ are the same, both associated with $\phi(N) = \left(1-q^N\right)/(1-q)$. Modification removes the symmetry. In the undeformed limit for any $\nu \in \mathbb{R}$, the Hamiltonians in (33) and (34) approach the classical expression $H = N + \frac{1}{2}(1-K)$ related (see [5]) to the Hamiltonian of the two-particle Calogero model. Consequently, we believe that the above discussion may throw light on possible integrable deformations of the Calogero model.

The (p,q)-oscillator algebra (3), with $\phi(N) = \left(q^N - p^{-N}\right)/\left(q-p^{-1}\right)$, has a $q \leftrightarrow p^{-1}$ symmetry. The symmetry noted above in connection with the unmodified limits of the Hamiltonians in (33) and (34) is only a special case of this. This symmetry is broken in the modification (4). Let us now consider a modification of the (p,q)-oscillator algebra (3) preserving the $q \leftrightarrow p^{-1}$ symmetry. To this end, we take

$$\begin{aligned} &\tilde a\tilde a^\dagger - q^{1+2\nu\tilde K}\tilde a^\dagger\tilde a = p^{-(\tilde N+\nu(1-\tilde K))}\left[1+2\nu\tilde K\right]_{p,q} \qquad \tilde K = (-1)^{\tilde N} \\ &\left[\tilde N, \tilde a\right] = -\tilde a \qquad \left[\tilde N, \tilde a^\dagger\right] = \tilde a^\dagger \end{aligned} \tag{35}$$

corresponding to $\phi(\tilde N) = \left[\tilde N + \nu\left(1-\tilde K\right)\right]_{p,q}$. This modification of (1) is actually the same as the (p,q)-paraboson algebra proposed in [7], though presented differently there.

Now, let

$$\mathcal{B}_0 = \tfrac{1}{2}(\tilde N + \nu + \tfrac{1}{2}) \qquad \mathcal{B}_+ = \frac{1}{[2]_{p,q}}\tilde a^{\dagger 2} \qquad \mathcal{B}_- = \frac{1}{[2]_{p,q}}\tilde a^2 \tag{36}$$

and

$$\mathcal{V}_+ = \frac{1}{\sqrt{[2]_{p,q}}}\tilde{a}^\dagger \qquad \mathcal{V}_- = \frac{1}{\sqrt{[2]_{p,q}}}\tilde{a}. \tag{37}$$

These operators can be seen to generate a (p, q)-deformation of the classical superalgebra $osp(1|2)$ as follows:

$$\left[\mathcal{B}_0, \mathcal{B}_\pm\right] = \pm\mathcal{B}_\pm \qquad \mathcal{B}_-\mathcal{B}_+ - (q/p)^2\mathcal{B}_+\mathcal{B}_- = [2\mathcal{B}_0]_{p^2,q^2} \tag{38}$$

and

$$\begin{gathered}
\left[\mathcal{B}_0, \mathcal{V}_\pm\right] = \pm\tfrac{1}{2}\mathcal{V}_\pm \qquad \left[\mathcal{B}_\pm, \mathcal{V}_\pm\right] = 0 \\
\mathcal{B}_\pm\mathcal{V}_\mp - (q/p)^{\mp 1}\mathcal{V}_\mp\mathcal{B}_\pm = [\mp 1]_{p,q}\left\{2\mathcal{B}_0 \mp \tfrac{1}{2} \pm \nu\tilde{K}\right\}_{p,q}\mathcal{V}_\pm \\
\mathcal{V}_-\mathcal{V}_+ + (q/p)^{\frac{1}{2}(1+2\nu\tilde{K})}\mathcal{V}_+\mathcal{V}_- = \frac{\left[1+2\nu\tilde{K}\right]_{p,q}[2\mathcal{B}_0]_{p,q}}{[2]_{p,q}\left[\frac{1}{2}(1+2\nu\tilde{K})\right]_{p,q}} \qquad \{\mathcal{V}_\pm, \mathcal{V}_\pm\} = \mathcal{B}_\pm \\
\text{with} \quad \{X\}_{p,q} = \frac{q^X + p^{-X}}{q = p^{-1}} \qquad \tilde{K} = (-1)^{2\mathcal{B}_0-\nu-1/2}.
\end{gathered} \tag{39}$$

The dynamical algebra of the (p, q)-paraboson (35), with $q \leftrightarrow p^{-1}$ symmetry, is another $U_{p^2,q^2}(su(1, 1) \oplus u(1))$ generated by $\{\mathcal{B}_0, \mathcal{B}_\pm\}$ defined above (36) and an additional central element, say $\mathcal{Z}$, exactly as in the case of the oscillator (4); but, in this case, the algebra of $\{\mathcal{B}_0, \mathcal{B}_\pm, \mathcal{Z}\}$ is independent of ν. (Note that equations (38) and (39) above are to replace equations (41) and (44) of [7] which contain some errors). To obtain the Fock representation, Bargmann–Fock realization and coherent states for the (p, q)-paraboson (35) one can follow the same procedure detailed above in the case of the modified (p, q)-oscillator (4). When $p = 1$ the algebra (35) corresponds to a q-paraboson algebra [17]. In the undeformed limit $(p = q = 1)$ (36)–(39) represent the usual paraboson realization of the classical $osp(1|2)$ (see, e.g. [18]).

Finally, let us note that the algebra (4) may be used to study a deformed centreless Virasoro algebra in which the operator K induces a doubling of the number of generators. Defining

$$L_n = p^N a^{\dagger n+1} a \qquad \tilde{L}_n = L_n K \qquad n \in \mathbb{Z} \tag{40}$$

one has, with $[X, Y]_{\alpha,\beta} = \alpha XY - \beta YX$,

$$\begin{aligned}
[L_n, L_m]_{p^{n-m},q^{m-n}} &= [m-n]_{p,q}L_{m+n} + 2\nu(-1)^n[m-n]_{-p,q}\tilde{L}_{m+n} \\
\left[\tilde{L}_n, \tilde{L}_m\right]_{(-1)^m p^{n-m},(-1)^n q^{m-n}} &= [m-n]_{p,q}L_{m+n} + 2\nu(-1)^n[m-n]_{-p,q}\tilde{L}_{m+n} \\
\left[\tilde{L}_n, L_m\right]_{(-1)^m p^{n-m},q^{m-n}} &= [m-n]_{p,q}\tilde{L}_{m+n} + 2\nu(-1)^n[m-n]_{-p,q}L_{m+n}.
\end{aligned} \tag{41}$$

In the limit $\nu = 0$ the algebra (41) agrees with the deformations of the Virasoro algebra known already [13, 19, 20]. The algebra (41) is a special case of the generalized Virasoro algebra considered earlier [21] as obtainable using any single-mode generalized oscillator algebra of the type (5). It is to be noted, however, that the closure property combined with the requirement of having c-number structure constants has enforced a doubling of the number of generators in the algebra (41). Having c-number structure constants may be of importance in constructing a generalized Jacobi identity [13, 20] for the deformed Virasoro generators. A deformed Virasoro algebra associated with the (p, q)-paraboson algebra (35) may also be studied along similar lines.

In (41) the generators $\{L_{\pm 1}, L_0, \tilde{L}_{\pm 1}, \tilde{L}_0\}$ constitute a subalgebra. In the undeformed limit ($p = q = 1$) the algebra (41) has an interesting structure:

$$
\begin{aligned}
&[L_n, L_m] = (m-n)L_{m+n} + \nu\left((-1)^n - (-1)^m\right)\tilde{L}_{n+m} \\
&\left[\tilde{L}_n, \tilde{L}_m\right]_{(-1)^m,(-1)^n} = (m-n)L_{m+n} + \nu\left((-1)^n - (-1)^m\right)\tilde{L}_{n+m} \\
&\left[\tilde{L}_n, L_m\right]_{(-1)^m,1} = (m-n)\tilde{L}_{m+n} + \nu\left((-1)^n - (-1)^m\right)L_{n+m}\,.
\end{aligned}
\tag{42}
$$

It is to be noted that the closure property of the algebra (42) demands that both the commutators and anticommutators appear. To understand the possible structure of the central term in (42) it may be fruitful to consider a Sugawara-type construction based on multimode paraboson algebra. We will study this topic elsewhere.

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